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STABLE STRUCTURES ON MANIFOLDS: II STABLE MANIFOLDS

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1. Introduction

The first paper [1] of this series studied the group of homeomorphisms of the n-sphere. The present paper is divided into four parts, as follows.

In the first part we consider the group of homeomorphisms of a connected topological manifold. The results obtained are weakened generalizations of the theorems proved in [1] for the n-sphere.

The second part introduces stable structures. The results of the first part can be improved to full generalizations of the theorems in [1] if and only if the manifold supports a stable structure.

The third part studies the relations between stable structures and covering spaces.

The fourth part considers the questions of existence and uniqueness of stable structures.

The machinery developed here will be applied in the following paper to some problems in the field of topological manifolds.

(I)

2. Definitions

The reader is first referred to the definitions supplied in [1, § 2].

The set of points $\{(x_1, \cdots, x_n) : \sum x_i^2 \leqq 1\}$ in euclidean n-space R^n will be denoted by D^n and its boundary by S^{n-1}. D^n and any space homeomorphic to D^n will be called a *closed n-cell*. S^{n-1} and any space homeomorphic to S^{n-1} will be called an $n-1$ *sphere*.

M^n will always denote a connected n-dimensional topological manifold, and $H(M^n)$ the group of homeomorphisms of M^n onto itself.

A k-manifold M^k in an n-manifold M^n will be called *locally flat* if each point of M^k has a neighborhood U in M^n such that the pair $(U, U \cap M^k)$ is topologically equivalent to the pair (R^n, R^k). An embedding $f: M^k \to M^n$

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will be called *locally flat* if $f(M^k)$ is locally flat in M^n. An embedding $f\colon D^n \to M^n$ will be called *locally flat* if f/S^{n-1} is locally flat.

Hom (D^n, M^n) will denote the set of all locally flat embeddings of D^n into M^n. If $h \in H(M^n)$ and $f \in \operatorname{Hom}(D^n, M^n)$, then $hf \in \operatorname{Hom}(D^n, M^n)$. Hence $H(M^n)$ acts as a transformation group on Hom (D^n, M^n).

3. Annular equivalence of embeddings of D^n in M^n

In a manner similar to that of [1], we will proceed by first studying Hom (D^n, M^n) and then using the information obtained to study $H(M^n)$.

Let f_0 and f_1 be elements of Hom (D^n, M^n) such that $f_0(D^n)$ lies in the interior of $f_1(D^n)$. If there is an embedding $F\colon S^{n-1} \times [0, 1] \to M^n$ such that, for all $x \in S^{n-1}$, $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$, then F will be called a *strict annular equivalence* between f_0 and f_1, and we write both

$$f_0 \underset{A}{\sim} f_1 \qquad \text{and} \qquad f_1 \underset{A}{\sim} f_0 \,.$$

Strict annular equivalence is not an equivalence relation, but induces one as follows. Two elements f and f' of Hom (D^n, M^n) will be called *annularly equivalent*, written

$$f \underset{a}{\sim} f' \,,$$

if there is a finite sequence of elements $f = f_0, f_1, \cdots, f_k = f'$ of Hom (D^n, M^n) such that $f_i \underset{A}{\sim} f_{i+1}$ for $i = 0, 1, \cdots, k-1$. Annular equivalence is an equivalence relation.

LEMMA 3.1. *Let f be an element of* Hom (D^n, M^n), *and U an open set in M^n. Then there is an element f' of* Hom (D^n, M^n) *such that $f'(D^n) \subset U$ and $f \underset{a}{\sim} f'$. If U meets $f(D^n)$, then we can make $f \underset{A}{\sim} f'$.*

If U meets $f(D^n)$, then U also meets the interior of $f(D^n)$. Let g be a homeomorphism of D^n onto $f(D^n)$ such that

(i) $g/S^{n-1} = f/S^{n-1}$

(ii) $g(0) \in U$.

Then there is a $t > 0$ such that the image under g of the n-cell D_t of radius t and center at the origin lies in U. If p denotes the radial contraction of D^n onto D_t, then $f' = gp$ is related to g, and hence to f, by a strict annular equivalence.

If U misses $f(D^n)$, let u be a fixed point of U. According to [2], there is an embedding $F\colon S^{n-1} \times [0, 1] \to M^n - \operatorname{Int} f(D^n)$ such that

(i) $F(x, 0) = f(x)$ for all $x \in S^{n-1}$.

If m is contained in $F(S^{n-1} \times (0, 1))$, there is a homeomorphism h of $M - \operatorname{Int} f(D^n)$ onto itself which restricts to the identity on $f(S^{n-1})$ and carries m onto u. Hence F may be chosen so that, in addition to (i), we have

(ii) $F(S^{n-1} \times (0, 1))$ contains u.

Let g be a homeomorphism of D^n onto $f(D^n) \cup F(S^{n-1} \times [0, 1])$ such that

(i) $g(x) = F(x, 1)$ for all $x \in S^{n-1}$

(ii) $g(0) = u$.

Then f' is obtained from g as in the first part of the proof, and $f \underset{A}{\sim} g \underset{A}{\sim} f'$.

As in [1], we now inquire how much the relation of annular equivalence generalizes that of strict annular equivalence.

LEMMA 3.2. *Let $f_1 \underset{A}{\sim} f_2 \underset{A}{\sim} f_3$ be elements of* Hom (D^n, M^n) *such that $f_i(D^n) \subset \operatorname{Int} f_j(D^n) \subset f_j(D^n) \subset \operatorname{Int} f_k(D^n)$ for some permutation (i, j, k) of $(1, 2, 3)$. Then $f_1 \underset{A}{\sim} f_3$.*

This follows directly from [1, Lemma 3.2].

LEMMA 3.3. *Let $f_1 \underset{A}{\sim} f_2 \underset{A}{\sim} f_3 \underset{A}{\sim} f_4 \underset{A}{\sim} f_5$, with $f_1(D^n)$, $f_3(D^n)$ and $f_5(D^n)$ mutually disjoint. Then there is a $g \in$* Hom (D^n, M^n) *such that $f_1 \underset{A}{\sim} g \underset{A}{\sim} f_5$.*

Picturing $h_5(D^n)$ as very small, there is a homeomorphism h of M^n such that

$$h/f_1(D^n) = 1 \qquad h/f_3(D^n) = 1 \qquad hf_2(D^n) \cap f_5(D^n) = \varnothing \ .$$

Since $hf_2 \underset{A}{\sim} hf_3 = f_3$, there is a homeomorphism h' of M^n such that

$$h'/f_3(D^n) = 1 \qquad h'/f_5(D^n) = 1 \qquad h'hf_2(D^n) \subset \operatorname{Int} f_4(D^n) \ ,$$

simply by shrinking $hf_2(D^n)$ close enough to $f_3(D^n)$.

Now let $g = h'^{-1}f_4$.

Then $g = h'^{-1}f_4 \underset{A}{\sim} h'^{-1}f_5 = f_5$.

Now $h'hf_2 \underset{A}{\sim} h'hf_3 = f_3 \underset{A}{\sim} f_4$, and $f_3(D^n) \subset \operatorname{Int} h'hf_2(D^n) \subset h'hf_2(D^n) \subset \operatorname{Int} f_4(D^n)$. So by Lemma 3.2, $h'hf_2 \underset{A}{\sim} f_4$.

But then $h'f_1 = h'hf_1 \underset{A}{\sim} h'hf_2 \underset{A}{\sim} f_4$, and $h'f_1(D^n) \subset \operatorname{Int} h'hf_2(D^n) \subset h'hf_2(D^n) \subset \operatorname{Int} f_4(D^n)$. So again by Lemma 3.2, $h'f_1 \underset{A}{\sim} f_4$.

Hence $f_1 \underset{A}{\sim} h'^{-1}f_4 = g \underset{A}{\sim} f_5$, as desired.

THEOREM 3.4. *Let f and f' be annularly equivalent elements of* Hom (D^n, M^n) *with disjoint images. Then there is a $g \in$* Hom (D^n, M^n) *such that*

$$f \underset{A}{\sim} g \underset{A}{\sim} f' \ .$$

Let $f = f_0 \underset{A}{\sim} f_1 \underset{A}{\sim} \cdots \underset{A}{\sim} f_k = f'$.

First we may assume that for no j do we have either $f_{j-1}(D^n) \subset f_j(D^n) \subset f_{j+1}(D^n)$ or the reverse, for otherwise by Lemma 3.2 we could drop f_j from the chain and write $f_{j-1} \underset{A}{\simeq} f_{j+1}$.

Second, we may assume that $f_0(D^n) \subset f_1(D^n)$. For if $f_0(D^n) \supset f_1(D^n)$, we can slightly expand f_0 to g_0 such that $f_0 \underset{A}{\simeq} g_0 \underset{A}{\simeq} f_1$ and $f_0(D^n) \subset g_0(D^n) \supset f_1(D^n)$, and then add g_0 to the chain between f_0 and f_1.

Similarly, we shall assume that $f_{k-1}(D^n) \supset f_k(D^n)$.

Thus we may write

$$f_0(D^n) \subset f_1(D^n) \supset f_2(D^n) \subset \cdots \subset f_{k-1}(D^n) \supset f_k(D^n) .$$

For each $j = 0, 1, \cdots, k/2$, let U_{2j} be a small open set in $\operatorname{Int} f_{2j}(D^n)$, chosen so that $U_{2i} \cap U_{2j} = \emptyset$ for $i \neq j$. For each such j, use Lemma 3.1 to obtain an element $f_{2j}^* \in \operatorname{Hom}(D^n, M^n)$ such that $f_{2j}^*(D^n) \subset U_{2j}$ and $f_{2j}^* \underset{A}{\simeq} f_{2j}$.

Then by Lemma 3.2,

$$f^* = f_0^* \underset{A}{\simeq} f_1 \underset{A}{\simeq} f_2^* \underset{A}{\simeq} \cdots \underset{A}{\simeq} f_{k-1} \underset{A}{\simeq} f_k^* = f'^* .$$

Repeated use of Lemma 3.3 proves the existence of a $g^* \in \operatorname{Hom}(D^n, M^n)$ such that

$$f^* = f_0^* \underset{A}{\simeq} g^* \underset{A}{\simeq} f_k^* = f'^* .$$

Since $f^* \underset{A}{\simeq} f$, $f'^* \underset{A}{\simeq} f'$ and f and f' have disjoint images, there is a homeomorphism h of M^n such that

$$hf^* = f \qquad \text{and} \qquad hf'^* = f' .$$

Then let $g = hg^*$, and the theorem is proved.

If $M^n = S^n$, the above theorem is a weakened version of [1, Theorem 3.5].

4. Stable homeomorphisms

Let h be a homeomorphism of M^n onto itself. If there is a non-empty open set $U \subset M^n$ such that $h/U = 1$, we will say that h is *somewhere the identity*. If there is a closed n-cell E with locally flat boundary in M^n such that $h/M^n - E = 1$, we will say that h is *almost everywhere the identity*.

$SH(M^n)$, the group of *stable homeomorphisms* of M^n, will consist of products of homeomorphisms, each of which is somewhere the identity. $SH_0(M^n)$ will consist of products of homeomorphisms, each of which is almost everywhere the identity.

It is shown in [3] that $SH_0(M^n)$ is the intersection of all non-trivial normal subgroups of $H(M^n)$, and is, furthermore, simple. In particular, $SH_0(M^n)$ must be arcwise connected in the compact-open topology. Hence every homeomorphism in $SH_0(M^n)$ is isotopic to the identity through homeomorphisms in $SH_0(M^n)$.

A set of homeomorphisms of the space X onto itself is called *complete* if every homeomorphism of X which agrees with some homeomorphism in the set in a neighborhood of each point of X is itself in the set. If $g \in H(M^n)$ agrees with $h \in SH(M^n)$ on the non-empty open set U, then $h^{-1}g/U = 1$. Then $h^{-1}g \in SH(M^n)$. Hence $g \in SH(M^n)$. Thus $SH(M^n)$ is certainly complete.

5. Stable equivalence of embeddings of D^n in M^n

Let f_1 and f_2 be elements of Hom (D^n, M^n). If there is a stable homeomorphism $h \in SH(M^n)$ such that $hf_1 = f_2$, then we say that f_1 and f_2 are *stably equivalent*, and write

$$f_1 \underset{s}{\sim} f_2 .$$

Stable equivalence is an equivalence relation, so Hom (D^n, M^n) divides up into stable equivalence classes. The set of these stable equivalence classes will be denoted by

$$\mathrm{Hom}_s\,(D^n, M^n) .$$

Let $f_1 \underset{s}{\sim} f_2$, and choose $h \in SH(M^n)$ so that $hf_1 = f_2$. If $g \in H(M^n)$, then $ghg^{-1} \in SH(M^n)$ because $SH(M^n)$ is normal in $H(M^n)$. Hence $(ghg^{-1})gf_1 = gf_2$, so $gf_1 \underset{s}{\sim} gf_2$. Thus $H(M^n)$ acts on Hom (D^n, M^n) by permuting the stable equivalence classes, and therefore induces an action of $H(M^n)$ on $\mathrm{Hom}_s\,(D^n, M^n)$.

LEMMA 5.1. *Let f_1 and f_2 be stably equivalent elements of* Hom(D^n, M^n) *and g a homeomorphism of M^n such that $gf_1 = f_2$. Then $g \in SH(M^n)$.*

For if h is a stable homeomorphism such that $hf_1 = f_2$, then h and g agree on the non-empty open set Int $f_1(D^n)$. Hence g must also be stable.

COROLLARY 1. *If an element of $H(M^n)$ leaves one stable equivalence class of* Hom (D^n, M^n) *fixed, it is an element of $SH(M^n)$, and therefore leaves all stable equivalence classes fixed.*

COROLLARY 2. *$H(M^n)/SH(M^n)$ acts as a regular permutation group on* $\mathrm{Hom}_s\,(D^n, M^n)$*, and is therefore in one-one correspondence with a subset of* $\mathrm{Hom}_s\,(D^n, M^n)$.

Notice that we do not assert that this action is transitive. Hence if $M^n = S^n$, the above Corollary is a weakened version of [1, Corollary 2 to Lemma 5.1].

THEOREM 5.2. *Let f and f' be elements of* Hom (D^n, M^n) *such that $f \underset{A}{\sim} f'$. Then there is a stable homeomorphism h of M^n such that $f =$*

hf'. *Furthermore, h may be selected from $SH_0(M^n)$.*

Assume that $f(D^n) \subset \operatorname{Int} f'(D^n)$, and let $F\colon S^{n-1} \times [0, 1] \to M^n$ be a strict annular equivalence between f and f'. According to [2], F may be extended to a locally flat embedding $F^*\colon S^{n-1} \times [0, 2] \to M^n$. Let D^* denote the closed n-cell in M^n with boundary $F^*(S^{n-1} \times 2)$ which contains $f(D^n)$ and $f'(D^n)$.

A homeomorphism $h \in SH_0(M^n)$ may then be constructed so that

(i) $hf'(x) = f(x)$ for all $x \in D^n$

(ii) $hF^*(x, t) = F^*(x, 2t - 2)$ for all $x \in S^{n-1}$ and all $t \in [1, 2]$

(iii) $h/M^n - D^* = 1$.

Then h is well-defined because, for all $x \in S^{n-1}$,

$$hf'(x) = hF^*(x, 1) = F^*(x, 0) = f(x)$$

and

$$hF^*(x, 2) = F^*(x, 2) .$$

COROLLARY. *If $f \underset{a}{\simeq} f'$, then there is an $h \in SH_0(M^n)$ such that $f = hf'$. Hence in particular, $f \underset{s}{\simeq} f'$.*

THEOREM 5.3. *Let h be a stable homeomorphism of M^n whose restriction to the non-empty open set U is the identity. If $f \in \operatorname{Hom}(D^n, M^n)$, then $f \underset{a}{\simeq} hf$.*

According to Lemma 3.1, there is an element f' of $\operatorname{Hom}(D^n, M^n)$ such that $f'(D^n) \subset U$ and $f' \underset{a}{\simeq} f$. Then clearly $hf' \underset{a}{\simeq} hf$. Therefore

$$f \underset{a}{\simeq} f' = hf' \underset{a}{\simeq} hf ,$$

hence

$$f \underset{a}{\simeq} hf .$$

COROLLARY. *If $f \underset{s}{\simeq} f'$, then $f \underset{a}{\simeq} f'$.*

Choose a stable homeomorphism h of M^n such that $f' = hf$. Write h as a product $h_k h_{k-1} \cdots h_2 h_1$ of homeomorphisms, each of which is somewhere the identity. Then by Theorem 5.3,

$$f \underset{a}{\simeq} h_1 f \underset{a}{\simeq} h_2 h_1 f \underset{a}{\simeq} \cdots \underset{a}{\simeq} h_k h_{k-1} \cdots h_2 h_1 f = hf = f' ,$$

hence

$$f \underset{a}{\simeq} f' .$$

Combining the Corollaries to Theorems 5.2 and 5.3, we get

THEOREM 5.4. *Two elements of* $\operatorname{Hom}(D^n, M^n)$ *are stably equivalent if and only if they are annularly equivalent.*

Note that we also obtain

THEOREM 5.5. *Let f and f' be stably equivalent elements of* $\operatorname{Hom}(D^n, M^n)$. *Then there is a homeomorphism $h \in SH_0(M^n)$ such that $f = hf'$.*

For if f and f' are stably equivalent, they are also annularly equivalent, and then such an h exists by the Corollary to Theorem 5.2.

We will generally prefer the name *stable equivalence* to *annular equivalence*, except when specifically referring to the results of § 3 and the present section.

6. The structure of $\operatorname{Hom}_s(D^n, M^n)$ and $SH(M^n)$

Let U be a non-empty connected open subset of M^n, and consider $\operatorname{Hom}(D^n, U)$. If i: $U \subset M^n$ is the inclusion, then the association of $f \in \operatorname{Hom}(D^n, U)$ with $i \cdot f \in \operatorname{Hom}(D^n, M^n)$ defines a map from $\operatorname{Hom}(D^n, U)$ to $\operatorname{Hom}(D^n, M^n)$. Since annular equivalence in U implies annular equivalence in M^n, we also get a map

$$i_*: \operatorname{Hom}_s(D^n, U) \rightarrow \operatorname{Hom}_s(D^n, M^n) .$$

This map is onto by Lemma 3.1, but is not in general one-one (suppose for example that M^n is non-orientable and U orientable).

If $M^n = S^n$, then it follows easily from [1, Theorem 3.5 (i)] that i_* is always one-one. Theorem 3.4 of the present paper is only a weakened generalization of this theorem, hence even for orientable M^n, we can not conclude that i_* is always one-one.

The following result, although not the best possible, will be sufficient for present purposes.

THEOREM 6.1. *Let E be a closed n-cell with locally flat boundary in M^n. Then i_*: $\operatorname{Hom}_s(D^n, M^n - E) \rightarrow \operatorname{Hom}_s(D^n, M^n)$ is one-one.*

Let $f, f' \in \operatorname{Hom}(D^n, M^n - E)$ be annularly equivalent in M^n. Using Lemma 3.1, we can assume that f and f' have disjoint images. Then by Theorem 3.4 there is an element $g \in \operatorname{Hom}(D^n, M^n)$ such that

$$f \underset{A}{\simeq} g \underset{A}{\simeq} f' .$$

Picturing E as very small, there is a homeomorphism h of M^n such that

$$h/f(D^n) = 1 \qquad h/f'(D^n) = 1 \qquad hg(D^n) \subset M^n - E .$$

Then $f = hf \underset{A}{\simeq} hg \underset{A}{\simeq} hf' = f'$ in $M^n - E$, so $f \underset{a}{\simeq} f'$ in $M^n - E$.

COROLLARY. *Let A be a closed subset of M^n which does not disconnect M^n and which is contained in a closed n-cell E with locally flat boundary* *in* M^n. *Then* i_*: $\mathrm{Hom}_s(D^n, M^n - A) \to \mathrm{Hom}_s(D^n, M^n)$ *is one-one.*

Let $j: (M^n - E) \subset (M^n - A)$ and $k: (M^n - E) \subset M^n$. Then

$$k_* = i_* j_* .$$

Then k_* is one-one by Theorem 6.1 and, since $M^n - A$ is connected, j_* is onto by Lemma 3.1. Hence i_* must be one-one.

THEOREM 6.2. *Let h be a stable homeomorphism of* M^n *and* E_1, E_2 *closed n-cells with locally flat boundaries in* M^n. *If* $E_1 \cup hE_1$ *is disjoint from* E_2, *then there is a stable homeomorphism* h' *of* M^n *which agrees with h on* E_1 *and whose restriction to* E_2 *is the identity.*

Let f be a homeomorphism of D^n onto E_1. Then f and hf are stably equivalent elements of $\mathrm{Hom}(D^n, M^n)$. By Theorem 6.1, f and hf must also be stably equivalent in $M^n - E_2$. By Theorem 5.5, there is a homeomorphism $h_0 \in SH_0(M^n - E_2)$ such that $h_0 f = hf$. Since h_0 must have compact support, h_0 can be extended over E_2 by the identity to yield h'.

COROLLARY. *Any stable homeomorphism of* M^n *can be written as the product of two homeomorphisms, each of which is somewhere the identity.*

Theorem 6.2 seems to be a word-for-word generalization of [1, Theorem 7.1], which was the principal structure theorem for stable homeomorphisms of S^n. Note however that in the case of S^n, the theorem can be reworded so that E_2, instead of being disjoint from $E_1 \cup hE_1$, contains $E_1 \cup hE_1$ in its interior. In the case of an arbitrary connected manifold M^n, the reworded theorem is no longer equivalent to the original one.

Note that in the following theorem, we are not free to choose E_2.

THEOREM 6.3. *Let h be a stable homeomorphism of* M^n *and* E_1 *a closed n-cell with locally flat boundary in* M^n *such that* E_1 *and* hE_1 *are disjoint. Then there exists a closed n-cell* E_2 *with locally flat boundary in* M^n *which contains* $E_1 \cup hE_1$ *in its interior, and a stable homeomorphism* h' *of* M^n *which agrees with h on* E_1 *and whose restriction to* $M^n - E_2$ *is the identity.*

Let f be a homeomorphism of D^n onto E_1. Then f and hf are annularly equivalent elements of $\mathrm{Hom}(D^n, M^n)$ by Theorem 5.4. Since they have disjoint images, Theorem 3.4 asserts the existence of an element $g \in \mathrm{Hom}(D^n, M^n)$ such that $f \underset{A}{\sim} g \underset{A}{\sim} hf$. Let $E_2 = g(D^n)$.

Let $F: S^{n-1} \times [0, 1] \to M^n$ be a strict annular equivalence between f and g, and $F^*: S^{n-1} \times [0, 1] \to M^n$ a strict annular equivalence between hf and g.

A homeomorphism h' may then be constructed so that

(i) $h'f(x) = hf(x)$ for all $x \in D^n$,
(ii) $h'F(x, t) = F^*(x, t)$ for all $x \in S^{n-1}$ and all $t \in [0, 1]$,
(iii) $h'/M^n - E_2 = 1$.
It is easily seen that h' is well-defined, and the theorem is proved.

THEOREM 6.4. *$SH(M^n)$ is the completion of $SH_0(M^n)$, and may therefore be characterized as the smallest non-trivial complete normal subgroup of $H(M^n)$.*

Let h be a stable homeomorphism of M^n and m a point of M^n. Let $f \in \text{Hom}(D^n, M^n)$ be chosen so that $f(0) = m$. Since f and hf are stably equivalent, there is, by Theorem 5.5, a homeomorphism $h_0 \in SH_0(M^n)$ such that $h_0 f = hf$. Then h and h_0 agree on a neighborhood of m, so $SH(M^n)$ is contained in the completion of $SH_0(M^n)$. Since $SH(M^n)$ is already complete, it is the completion of $SH_0(M^n)$.

The characterization of $SH(M^n)$ then follows from the fact that $SH_0(M^n)$ is the smallest non-trivial normal subgroup of $H(M^n)$.

THEOREM 6.5. *Let U be a connected open subset of M^n, and h a homeomorphism of M^n which takes U onto itself. If h/U is stable, then so is h stable.*

Let $f \in \text{Hom}(D^n, U)$. Then $f \underset{s}{\sim} hf$ in U. By Theorem 5.4, $f \underset{a}{\sim} hf$ in U. Then certainly $f \underset{a}{\sim} hf$ in M^n. Again by Theorem 5.4, $f \underset{s}{\sim} hf$ in M^n. By Lemma 5.1, h must be stable.

(II)

7. The pseudogroup of stable coordinate transformations

Let f_i be a homeomorphism of the open set $U_i \subset R^n$ onto the open set $V_i \subset R^n$ for $i = 1, 2$. If $V_1 \cap U_2$ is not empty, define the composition $f_2 f_1$ to be the map

$$(f_2/V_1 \cap U_2)\cdot(f_1/f_1^{-1}(V_1 \cap U_2)) : f_1^{-1}(V_1 \cap U_2) \to f_2(V_1 \cap U_2) .$$

If $V_1 \cap U_2$ is empty, the composition $f_2 f_1$ will not be defined.

Let P be a collection of homeomorphisms of open sets in R^n onto open sets in R^n. P will be called a *pseudogroup* provided

(i) the identity map of R^n is in P,
(ii) if $f, g \in P$, then $fg \in P$ whenever defined,
(iii) if $f \in P$, then $f^{-1} \in P$,
(iv) if $f\colon U \to V$ is in P and $U' \subset U$, then f/U' is in P,
(v) if $f\colon U \to V$ is a homeomorphism, and for each $x \in U$ there is a neighborhood U_x of x such that f/U_x is in P, then f is in P. (This is the *completeness* condition.)

A homeomorphism $f: U \to V$ is *stable at* $x \in U$ if there is a neighborhood U_x of x and a stable homeomorphism h of R^n onto itself such that

$$f/U_x = h/U_x .$$

Then f is *stable* if it is stable at each point of U. Note that by definition, the set of points of U at which f is stable is open.

Suppose now that $f: U \to V$ is *not* stable at $x \in U$. Let E be a closed n-cell with locally flat boundary, which contains x in its interior and which is itself contained in U. According to [2] and [4], f/E can be extended to a homeomorphism h of R^n onto itself. Now if f were stable at any point of the interior of E, then h would agree with a stable homeomorphism on some open set and would therefore itself be stable. But then f would be stable at x. Hence f cannot be stable at any point of the interior of E, so the set of points of U at which f is not stable is also open.

We have therefore proved

THEOREM 7.1. *If $f: U \to V$ is stable at $x \in U$, then it is stable on the component of U containing x. Similarly, if f is not stable at $x \in U$, it is not stable at any point of the component of U containing x.*

A homeomorphism $f: U \to V$ which is stable on U will be called a *stable coordinate transformation*. The collection of all stable coordinate transformations forms a pseudogroup, which we denote by $SP(R^n)$.

If $f: U \to V$ and h is a homeomorphism of R^n, then $hfh^{-1}: hU \to hV$. A pseudogroup P is called *normal* if $f \in P$ implies $hfh^{-1} \in P$ for every homeomorphism h of R^n. P is called *simple* if every pseudogroup contained in P which is invariant under conjugation by those homeomorphisms of R^n lying in P is either the trivial pseudogroup or else all of P.

THEOREM 7.2. *$SP(R^n)$ is the intersection of all non-trivial normal pseudogroups in R^n and is, furthermore, simple.*

The theorem will be proved by showing that any normal pseudogroup other than the trivial one must contain a homeomorphism, other than the identity, whose domain and range is all of R^n. Then since $SH_0(R^n)$ is the smallest non-trivial normal subgroup of $H(R^n)$, any non-trivial normal pseudogroup will have to contain $SH_0(R^n)$. Then by completeness it will also contain $SH(R^n)$. Finally by restriction it must contain $SP(R^n)$. Since all conjugations will be by stable homeomorphisms, we will at the same time have demonstrated the simplicity of $SP(R^n)$.

Suppose then that P is a non-trivial normal pseudogroup. We must show that P contains a homeomorphism of R^n onto itself. Start with an element $f\colon U \to V$ of P other than the identity. By restriction, if necessary, we can assume that U and V are disjoint. Let $x \in U$ and let U_0 be a small n-cell neighborhood of x whose closure lies in U. Then $V_0 = f(U_0)$ is a small n-cell neighborhood of $f(x)$ whose closure lies in V.

Now let g be a homeomorphism of R^n onto itself which restricts to the identity outside $U_0 \cup V_0$ such that

$$gf(x) \neq fg(x) .$$

Note that g is stable. Since P is normal, P must also contain $gfg^{-1}\colon U \to V$, and hence also $f^{-1}gfg^{-1}\colon U \to U$. Note that

$$f^{-1}gfg^{-1}\big(g(x)\big) = f^{-1}gf(x) \neq f^{-1}fg(x) = g(x) .$$

Hence $f^{-1}gfg^{-1}$ is not the identity. On the other hand, $f^{-1}gfg^{-1}$ restricts to the identity on $U - U_0$. Let h be the homeomorphism of R^n onto itself which agrees with $f^{-1}gfg^{-1}$ on U and the identity on $R^n - U_0$. Then h must lie in P by completeness, and the theorem is proved.

8. The sheaf of germs of stable structures

Let M^n be a connected topological n-manifold, and h a homeomorphism from an open set in R^n onto an open set $U \subset M^n$. We call h a *coordinate homeomorphism*, U a *coordinate neighborhood*, and the pair (U, h) a *local coordinate system*. If $x \in U$, the triple (x, U, h) will be called a *local coordinate system at* x.

Two triples (x, U, h) and (x', U', h') will be said to be *stably equivalent* if
(i) $x = x'$,
(ii) $h'^{-1}h\colon h^{-1}(U \cap U') \to h'^{-1}(U \cap U')$ is stable at $h^{-1}(x)$.
In such a situation we will also say that the local coordinate systems (U, h) and (U', h') are *stably equivalent at* x.

The stable equivalence class determined by a triple (x, U, h) will be denoted by $[x, U, h]$. The set of all such stable equivalence classes will be denoted by $S = S(M^n)$, and called the *sheaf of germs of stable structures on* M^n. A map

$$p\colon S(M^n) \to M^n$$

is defined by sending $[x, U, h]$ onto x. Then $p^{-1}(x) = S(x)$ is called the *stalk over* x.

If (U, h) is a local coordinate system, then with each $x \in U$ we may associate the element $[x, U, h]$ of $S(x)$, thus obtaining a section

$$\bar{h}\colon U \to S$$

of the sheaf over U.

As is customary, S is topologized with the maximal topology making all such sections continuous. In this topology, a set in S is open if and only if its inverse image under every $\bar{h}$ is open in M^n.

A map $f\colon S \to X$ is continuous if and only if $f\bar{h}\colon U \to X$ is continuous for all local coordinate systems (U, h). Then $p\colon S \to M^n$ is certainly continuous.

THEOREM 8.1. *Let (U, h) be a local coordinate system in M^n. Then the section $\bar{h}\colon U \to S$ is an open map.*

It is clearly sufficient to show that $\bar{h}(U)$ is open in S. For if U' is open in U, then $\bar{h}(U') = \overline{(h/h^{-1}U')}(U')$.

To show that $\bar{h}(U)$ is open in S, we must show that $(\bar{g})^{-1}\bar{h}(U)$ is open in M^n for any local coordinate system (V, g). Now $x \in (\bar{g})^{-1}\bar{h}(U)$ if and only if $x \in U \cap V$ and (U, h) is stably equivalent to (V, g) at x. But if (U, h) and (V, g) are stably equivalent at x, they are stably equivalent in a neighborhood of x, hence $(\bar{g})^{-1}\bar{h}(U)$ is open in M^n, and therefore $\bar{h}(U)$ is open in S.

9. Construction of S'

In this section we will construct over M^n a principal bundle S' with group and fibre the discrete group $H(R^n)/SH(R^n)$. The most important property of S will then be displayed in the next section by showing that S and S' are equivalent over M^n.

Let $(U_i, h_i)_I$ be a family of local coordinate systems on M^n such that $\bigcup_I U_i = M^n$. We will define, according to [5], a system of coordinate transformations in M^n with values in $H(R^n)/SH(R^n)$. If $h \in H(R^n)$, it will be convenient to denote the coset $h \cdot SH(R^n)$, which is an element of $H(R^n)/SH(R^n)$, by $[h]$.

If $x \in U_i \cap U_j$, let h be a homeomorphism of R^n which agrees with $h_j^{-1}h_i$ in a neighborhood of $h_i^{-1}(x)$. Then $[h]$ depends only on (U_i, h_i), (U_j, h_j) and x. The map

$$g_{ji}\colon U_i \cap U_j \to H(R^n)/SH(R^n)$$

is then defined by

$$g_{ji}(x) = [h]\ .$$

The map g_{ji} is constant on components of $U_i \cap U_j$ and hence continuous. Furthermore,

$$g_{kj}(x) \cdot g_{ji}(x) = g_{ki}(x) \qquad \text{for } x \in U_i \cap U_j \cap U_k\ ,$$

by the normality of $SH(R^n)$ in $H(R^n)$.

Theorem 3.2 of [5] then asserts the existence and uniqueness (up to bundle equivalence) of a principal bundle S' over M^n with group and fibre $H(R^n)/SH(R^n)$ and coordinate transformations g_{ji}.

A model for S' is constructed as follows. Let $T \subset M^n \times (H(R^n)/SH(R^n)) \times I$ be the set of triples $(x, [h], i)$ such that $x \in U_i$, where both $H(R^n)/SH(R^n)$ and I have the discrete topology. Then T is the union of the disjoint open subsets $U_i \times (H(R^n)/SH(R^n)) \times i$.

Two triples $(x, [h], i)$ and $(x', [h'], j)$ will be said to be *equivalent* if

(i) $x = x'$,

(ii) $g_{ji}(x)\cdot[h] = [h']$.

The equivalence class of $(x, [h], i)$ will be written $[x, [h], i]$, and the set of such equivalence classes, with the decomposition topology, will be denoted by S'. A map

$$p'\colon S' \to M^n$$

is defined by

$$p'([x, [h], i]) = x \ .$$

10. Equivalence of S and S'

We first define a map

$$\Phi\colon T \to S$$

by

$$\Phi((x, [h], i)) = [x, U_i, h_i h] \ .$$

If $[h] = [h']$, then $(h_i h')^{-1}(h_i h) = h'^{-1}h$ is certainly stable at $(h_i h)^{-1}(x)$. Hence $[x, U_i, h_i h] = [x, U_i, h_i h']$, and Φ is well-defined.

If $(x, [h], i)$ and $(x, [h'], j)$ are equivalent in T, then

$$[h'] = g_{ji}(x)\cdot[h] \ .$$

Then $\Phi((x, [h'], j)) = [x, U_j, h_j h']$. In a neighborhood of $(h_i h)^{-1}(x)$,

$$(h_j h')^{-1}(h_i h) = h^{-1}h_i^{-1}h_j h_j^{-1}h_i h = 1 \ ,$$

hence $[x, U_j, h_j h'] = [x, U_i, h_i h] = \Phi((x, [h], i))$.

Therefore Φ induces a map

$$\varphi\colon S' \to S \ .$$

THEOREM 10.1. *$\varphi\colon S' \to S$ is a homeomorphism such that $p\varphi = p'$.*

$$p\varphi([x, [h], i]) = p([x, U_i, h_i h]) = x = p'([x, [h], i]) \ .$$

To show that φ is a homeomorphism, we will show that

(i) Φ identifies two elements of T only if they are equivalent in T,

(ii) Φ is onto,

(iii) Φ is continuous,

(iv) Φ is open.

PROOF OF (i). If $\Phi((x, [h], i)) = \Phi((x', [h'], j))$, then $x = x'$ and $(h_j h')^{-1}(h_i h)$ is stable at $(h_i h)^{-1}(x)$. Then $[h'] = g_{ji}(x) \cdot [h]$, so $(x, [h], i)$ is equivalent to $(x', [h'], j)$ in T.

PROOF OF (ii). Let $[x, U, h] \in S$. Choose $i \in I$ so that $x \in U_i$. Let $h' \in H(R^n)$ agree with $h_i^{-1}h$ in a neighborhood of $h^{-1}(x)$. Then $[x, U, h] = [x, U_i, h_i h'] = \Phi((x, [h'], i))$.

PROOF OF (iii). For fixed $i \in I$ and $h \in H(R^n)$, $\Phi((x, [h], i)) = [x, U_i, h_i h]$ varies continuously with x by the very choice of topology for S. Since I and $H(R^n)/SH(R^n)$ have discrete topologies, Φ must be continuous.

PROOF OF (iv). Let U be open in U_i, and $i \in I$ and $h \in H(R^n)$ fixed. Then $\Phi(U \times [h] \times i) = (\overline{h_i h})(U)$ is open in S by Theorem 8.1. Since I and $H(R^n)/SH(R^n)$ have discrete topologies, Φ must be open, and the proof is completed.

Thus the sheaf $S(M^n)$ of germs of stable structures on M^n is a principal bundle with group and fibre the discrete group $H(R^n)/SH(R^n)$. Such an object differs from a regular covering space over M^n only in that $S(M^n)$ is not necessarily connected. However, as a principal bundle, the various components of $S(M^n)$ are equivalent over M^n, and in this generalized sense we state

THEOREM 10.2. *The sheaf $S(M^n)$ of germs of stable structures on M^n is a regular covering space over M^n.*

The well-defined normal subgroup of $\pi_1(M^n)$ corresponding to this regular covering will be called the *stability subgroup* of $\pi_1(M^n)$, and denoted by $S\pi_1(M^n)$.

If $f: M^n \to S(M^n)$ is a global section of $S(M^n)$, we call f a *stable structure* on M^n, say that M^n *admits* or *supports* a stable structure, and call M^n a *stable manifold*. In such a case, $S(M^n)$ would be homeomorphic to $M^n \times H(R^n)/SH(R^n)$.

Note that if U is open in M^n, $p^{-1}(U)$ is a copy of the sheaf of germs of stable structures on U. Hence every open subset of a stable manifold is stable.

Since $S(M^n)$ is a covering space over M^n, we have

THEOREM 10.3. *Every simply connected manifold admits a stable structure.*

It is clear that M^n admits a stable structure if and only if there exists a family $(U_i, h_i)_I$ of local coordinate systems such that $M^n = \bigcup_I U_i$ and $h_j^{-1}h_i$ is stable on $h_i^{-1}(U_i \cap U_j)$ for all $i, j \in I$. Such a family will be called a *stable coordinate system* on M^n.

Note that stable coordinate transformations are always orientation preserving. Hence a stable manifold must be orientable.

From [1, Ths. 11.1 and 11.2] we now have

THEOREM 10.4. *Every orientable differentiable or piecewise linear manifold admits a stable structure.*

11. Stable coordinate systems

THEOREM 11.1. *Let M^n be a connected topological manifold, $(f_\alpha)_A$ a collection of elements of* Hom (D^n, M^n) *and $(U_i)_I$ a collection of open n-cells covering M^n such that*

(i) *if $f_\alpha(D^n) \cup f_\beta(D^n) \subset U_i$, then $f_\alpha \underset{s}{\sim} f_\beta$ in U_i,*

(ii) *if V is a component of $U_i \cap U_j$, then there is an $\alpha \in A$ such that $f_\alpha(D^n) \subset V$.*

Then M^n supports a stable structure.

Let P be the set of pairs (α, i) such that $f_\alpha(D^n) \subset U_i$. According to [2] and [4], there is a homeomorphism h_α: $R^n \rightarrow U_i$ such that $h_\alpha / D^n = f_\alpha$. Then the family $(U_i, h_\alpha)_P$ is a family of local coordinate systems covering M^n. We claim it is a stable coordinate system on M^n.

We must show that $h_\beta^{-1}h_\alpha$ is stable on $h_\alpha^{-1}(U_i \cap U_j)$ for any two elements (α, i) and (β, j) of P. By Theorem 7.1, it will be sufficient to show that $h_\beta^{-1}h_\alpha$ is stable at at least one point from each component of $h_\alpha^{-1}(U_i \cap U_j)$.

If W is a component of $h_\alpha^{-1}(U_i \cap U_j)$, then $V = h_\alpha(W)$ is a component of $U_i \cap U_j$. According to (ii) above, there is a $\gamma \in A$ such that $f_\gamma(D^n) \subset V$. By (i), $f_\alpha \underset{s}{\sim} f_\gamma$ in U_i and $f_\beta \underset{s}{\sim} f_\gamma$ in U_j. We will show that $h_\beta^{-1}h_\alpha$ is stable at $h_\alpha^{-1}f_\gamma(0)$.

Since $f_\alpha \underset{s}{\sim} f_\gamma$ in U_i, there is a stable homeomorphism φ_i of U_i onto itself such that $\varphi_i f_\alpha = f_\gamma$. Then $\psi_i = h_\alpha^{-1}\varphi_i h_\alpha$ is a stable homeomorphism of R^n onto itself.

Similarly, there is a stable homeomorphism φ_j of U_j onto itself such that $\varphi_j f_\beta = f_\gamma$, and hence $\psi_j = h_\beta^{-1}\varphi_j h_\beta$ is a stable homeomorphism of R^n onto itself.

Finally, in a neighborhood of $h_\alpha^{-1}f_r(0)$, we have

$$h_\beta^{-1}h_\alpha = (h_\beta^{-1}f_\gamma)(f_\gamma^{-1}h_\alpha) = (h_\beta^{-1}\varphi_j h_\beta)(h_\alpha^{-1}\varphi_i^{-1}h_\alpha) = \psi_j\psi_i^{-1} ,$$

which is stable.

Thus $(U_i, h_\alpha)_P$ is a stable coordinate system on M^n.

A collection $(f_\alpha)_A$ of elements of Hom(D^n, M^n) will be said to be *stably dense* in M^n provided

(i) if $f_\alpha(D^n)$ and $f_\beta(D^n)$ both lie in the connected open subset U of M^n, then $f_\alpha \underset{s}{\sim} f_\beta$ in U

(ii) for each open set $U \subset M^n$ there is an $\alpha \in A$ such that $f_\alpha(D^n) \subset U$.

Then from Theorem 11.1 we have immediately the

COROLLARY. *If* Hom(D^n, M^n) *has a stable dense subset, then M^n supports a stable structure.*

THEOREM 11.2. *Every orientable triangulable manifold admits a stable structure.*

Let h be a fixed homeomorphism of D^n onto a closed n-simplex, Δ^n. Orient Δ^n and triangulate and orient M^n. Let $(f_\alpha)_A$ be the set of all orientation preserving simplicial embeddings of Δ^n into the interior of any n-simplex in M^n. Then we claim that $(f_\alpha h)_A$ is a stably dense subset of Hom(D^n, M^n).

Since the embeddings are into the *interiors* of simplexes, it is clear that each f_α, and hence each $f_\alpha h$, is locally flat. Condition (ii) above is immediate. To verify condition (i), note that a connected open subset of M^n is always $n-(n-1)$ connected, i.e., not disconnected by removal of the $n-2$ skeleton of M^n. Then if $f_\alpha(\Delta^n)$ and $f_\beta(\Delta^n)$ both lie in the connected open subset U of M^n, $f_\alpha(\Delta^n)$ can be pushed from n-simplex through $n-1$ simplex to n-simplex of U until it coincides with $f_\beta(\Delta^n)$. Thus (i) is also satisfied.

12. Stable atlases

The set of all stable coordinate systems on the stable manifold M^n may be partially ordered by inclusion. Since the union of an ascending chain of stable coordinate systems is again a stable coordinate system, Zorn's lemma applies, and there must exist a maximal stable coordinate system which we call a *stable atlas*.

If we were dealing with continuous structures instead of stable structures, there would be only one continuous atlas for M^n, which would contain all local coordinate systems on M^n. A stable atlas will not, of course, contain as many coordinate homeomorphisms, but the following theorem shows that it will contain all possible coordinate neighborhoods.

THEOREM 12.1. *Let M^n be a stable manifold and $(U_i, h_i)_I$ a stable atlas. If $U \subset M^n$ is an open set which can be embedded in R^n, then there is an open set $W \subset R^n$ and a homeomorphism h: $W \to U$ such that $(U, h) =$*

(U_i, h_i) for some $i \in I$. If U is homeomorphic to R^n, W can be chosen to be R^n.

First suppose that U is connected, and let $h'\colon W' \to U$ be a homeomorphism of an open set in R^n onto U. Let x be a point of U and $y' = h'^{-1}(x)$. Let U_j be a coordinate neighborhood of x in M^n, and $h_j\colon W_j \to U_j$ the associated coordinate homeomorphism. Let $y_j = h_j^{-1}(x)$.

According to [2] and [4], there is a homeomorphism g of R^n onto itself which agrees with $h'^{-1}h_j$ in a neighborhood of y_j. Let $W = g^{-1}(W')$ and $h = h'g/W$.

To show that (U, h) is included in the stable atlas, note that $h = h_j$ in a neighborhood of $y_j = h^{-1}(x)$. Then (U, h) is certainly stably equivalent to all members of the atlas at x. Since U is connected, Theorem 7.1 implies that (U, h) is stably equivalent to all members of the atlas at every point of U. By maximality of an atlas, $(U, h) = (U_i, h_i)$ for some $i \in I$.

Note that if U is homeomorphic to R^n, choosing $W' = R^n$ insures $W = R^n$.

U, if disconnected, can have at most countably many components. Divide R^n into countably many disjoint open compartments, and modify the above construction by stably shrinking W into one of the compartments. Applying the modified construction with a new compartment for each component of U proves the theorem.

If $(U_i, h_i)_I$ is a stable atlas for M^n and h a homeomorphism of R^n onto itself, then $(U_i, h_ih)_I$ is also a stable atlas for M^n, and coincides with the original one if and only if h is stable. Since M^n is connected, the technique of the above theorem indicates that all stable atlases for M^n may be obtained from a single one in this manner. Hence a stable manifold admits as many distinct stable atlases as there are elements of $H(R^n)/SH(R^n)$. This is, of course, in agreement with the fact that $S(M^n)$ is homeomorphic to $M^n \times H(R^n)/SH(R^n)$ whenever M^n is stable.

THEOREM 12.2. *Let U and V be connected open subsets of M^n such that*

(i) $M^n = U \cup V$,

(ii) *$U \cap V$ is connected.*

Then if U and V are stable, so is M^n.

Let $(U_i, h_i)_I$ and $(V_j, g_j)_J$ be stable coordinate systems for U and V, respectively. Let x be a point of $U \cap V$, and as in the preceding theorem, choose a homeomorphism h of R^n onto itself such that $(U_i, h_ih)_I$ and $(V_j, g_j)_J$ are stably equivalent at x. Since $U \cap V$ is connected, the two coordinate systems will be stably equivalent at every point of $U \cap V$.

Then $(U_i, h_i h)_I \cup (V_j, g_j)_J$ will be a stable coordinate system for M^n.

13. Stable homeomorphisms of stable manifolds

Let M^n and $M^{n\prime}$ be stable manifolds with stable coordinate systems $(U_i, h_i)_I$ and $(U'_j, h'_j)_J$, respectively. Let f: $M^n \to M^{n\prime}$ be a homeomorphism and x a point of M^n. Choose coordinate neighborhoods U_i and U'_j such that $x \in U_i$ and $f(x) \in U'_j$. Then we will say that f is *stable at* x if $h'^{-1}_j f h_i$ is stable at $h_i^{-1}(x)$. The definition is independent of the choice of local coordinate systems from the given stable coordinate systems. Since M^n is connected, it follows from Theorem 7.1 that, if f is stable at one point of M^n, it is then stable at every point of M^n, in which case we call f a *stable homeomorphism*. $SH(M^n)$ will denote the group of stable homeomorphisms of the stable manifold M^n onto itself.

This terminology and notation has already been used in a different sense in § 4. The following theorem removes any possibility of confusion.

THEOREM 13.1. *A homeomorphism h of the stable manifold M^n onto itself is stable in the new sense if and only if it is stable in the old sense. In particular, the stability of h is independent of the particular stable structure on M^n.*

If a homeomorphism h of M^n onto itself restricts to the identity on the non-empty open set U, then h is certainly stable in the new sense at points of U, and hence stable in the new sense on M^n. Then a product of such homeomorphisms must also be stable in the new sense. Thus stability in the old sense implies stability in the new sense.

If h is stable in the new sense, choose $x \in M^n$ and let $x' = h(x)$. Let U be an open n-cell in M^n containing both x and x', and g: $R^n \to U$ a homeomorphism such that (U, g) is contained in the stable atlas for M^n, according to Theorem 12.1. Then $g^{-1}hg$ is stable at $g^{-1}(x)$. By Theorem 6.4, there is a homeomorphism $h' \in SH_0(R^n)$ which agrees with $g^{-1}hg$ in a neighborhood of $g^{-1}(x)$. Then $gh'g^{-1}$ is a homeomorphism of U onto itself which agrees with h in a neighborhood of x, and which restricts to the identity near the boundary of U. Extend $gh'g^{-1}$ over M^n *via* the identity to obtain a homeomorphism h_1 of M^n.

Since h_1 agrees with h in a neighborhood of x, $h_2 = hh_1^{-1}$ restricts to the identity in a neighborhood of x. Then writing

$$h = h_2 h_1$$

expresses h as the product of two homeomorphisms, each of which is somewhere the identity. Hence stability in the new sense implies stability in the old sense.

14. The structure of $\mathrm{Hom}_s(D^n, M^n)$ and $SH(M^n)$ for stable manifolds

The following theorem shows that the results of §§ 3 and 6 can be improved to full generalizations of the theorems in [1] if and only if M^n supports a stable structure.

THEOREM 14.1. *Let M^n be a connected topological manifold. Then the following are equivalent:*

(i) *M^n is stable.*

(ii) *If h is a stable homeomorphism of M^n and E_1 and E_2 are closed n-cells with locally flat boundaries in M^n such that $E_1 \cup hE_1 \subset E_2$, then there is a stable homeomorphism h' of M^n which agrees with h on E_1 and whose restriction to $M^n - E_2$ is the identity.*

(iii) *If f and f' are annularly equivalent elements of* $\mathrm{Hom}(D^n, M^n)$ *such that $f(D^n) \subset \mathrm{Int}\, f'(D^n)$, then $f \underset{A}{\sim} f'$.*

(iv) *If U is a connected open subset of M^n, then i_*:* $\mathrm{Hom}_s(D^n, U) \to \mathrm{Hom}_s(D^n, M^n)$ *is one-one and onto.*

Proof that (i) *implies* (ii). Let $U = \mathrm{Int}\, E_2$ and let $g: R^n \to U$ be a coordinate homeomorphism chosen so that (U, g) is contained in the stable atlas for M^n, according to Theorem 12.1. Since h is stable, it follows from [2] and [4] that $g^{-1}hg/g^{-1}(E_1)$ extends to a stable homeomorphism h^* of R^n onto itself. According to [1, Th. 7.1], h^* can be chosen to lie in $SH_0(R^n)$. Then $gh^*g^{-1}: U \to U$ restricts to the identity near the boundary of U, and h' may be defined to be gh^*g^{-1} on U and the identity on $M^n - U$.

Proof that (ii) *implies* (iii). According to Theorem 5.4, there is a stable homeomorphism h of M^n such that $hf = f'$. Let $E_1 = f(D^n)$ and let E_2 be a closed n-cell with locally flat boundary containing $hE_1 = f'(D^n)$ in its interior. There is, by (ii), a homeomorphism h' of M^n which agrees with h on E_1 and restricts to the identity on $M^n - E_2$. Thus f and f' are stably equivalent, and hence annularly equivalent, in $\mathrm{Int}\, E_2$. Then $f \underset{A}{\sim} f'$ by [1, Th. 3.5 (i)].

Proof that (iii) *implies* (iv). By Lemma 3.1, i_* is onto. Let $f, f' \in \mathrm{Hom}(D^n, U)$ be annularly equivalent in M^n. Using Lemma 3.1, we can assume that $f(D^n) \subset \mathrm{Int}\, f'(D^n)$. Then $f \underset{A}{\sim} f'$ by (iii), so f and f' are annularly equivalent in U and i_* is one-one.

Proof that (iv) *implies* (i). Let f be a fixed element of $\mathrm{Hom}(D^n, M^n)$, and $(f_\alpha)_A$ the collection of all elements of $\mathrm{Hom}(D^n, M^n)$ which are stably equivalent to f. By Lemma 3.1 and (iv), $(f_\alpha)_A$ is stably dense in M^n. Then M^n supports a stable structure by the Corollary to Theorem 11.1.

(III)

15. Stable transportation of local coordinate systems

In this section we describe a procedure for recognizing whether or not a closed curve in M^n represents an element of the stability subgroup $S\pi_1(M^n)$.

Let α be a path in M^n from a to b, and

$$a = a_0, a_1, \cdots, a_k = b$$

a partition of α. For each $i = 1, \cdots, k$, let (U_i, h_i) be a local coordinate system in M^n such that

(i) the sub-path of α from a_{i-1} to a_i lies in U_i,

(ii) $h_{i+1}^{-1}h_i$ is stable at $h_i^{-1}(a_i)$ for $i = 1, \cdots, k-1$.

Then we will say that the local coordinate system at b, (b, U_k, h_k), is obtained from the local coordinate system at a, (a, U_1, h_1), by *stable transportation along the path* α.

LEMMA 15.1. *Let the local coordinate systems at* a, (a, U_1, h_1) *and* (a, U_1', h_1'), *be stably equivalent, and let the local coordinate systems at* b, (b, U_j, h_j) *and* (b, U_k', h_k'), *be obtained from those at* a *by stable transportation along the path* α. *Then* (b, U_j, h_j) *and* (b, U_k', h_k') *are stably equivalent.*

By taking a common refinement of the two partitions of α, we may assume that the same partition is used for both transportations.

If (U_i, h_i) and (U_i', h_i') are stably equivalent at a_{i-1}, they are stably equivalent at every point of the component of $U_i \cap U_i'$ containing a_{i-1}. Since both U_i and U_i' contain the sub-path of α from a_{i-1} to a_i, (U_i, h_i) and (U_i', h_i') are also stably equivalent at a_i. But (U_{i+1}, h_{i+1}) is stably equivalent to (U_i, h_i) at a_i, and (U_{i+1}', h_{i+1}') is stably equivalent to (U_i', h_i') at a_i. Thus (U_{i+1}', h_{i+1}') is stably equivalent to (U_{i+1}, h_{i+1}) at a_i, and the lemma is then proved inductively.

It is clear that a given local coordinate system can be stably transported along a given path which begins within its range. Then the above lemma implies that, if $[x, U, h]$ is an element of $S(M^n)$ and α a path in M^n from x to x', there is a unique element $[x', U', h']$ of $S(M^n)$ obtained from $[x, U, h]$ by stable transportation along α. Since $S(M^n)$ is a covering space over M^n, this element is the same as that obtained by covering α by a path in $S(M^n)$ beginning at $[x, U, h]$.

We therefore have

THEOREM 15.2. *Let* α *be a closed curve in* M^n. *Then* α *represents an element of the stability subgroup* $S\pi_1(M^n)$ *if and only if stable trans-*

portation of any local coordinate system around α produces a stably equivalent local coordinate system. Furthermore, since $S(M^n)$ is a regular covering space over M^n, if a single local coordinate system transports around α to a stably equivalent one, then so do all local coordinate systems.

Now let

$$f_0 \underset{A}{\sim} f_1 \underset{A}{\sim} \cdots \underset{A}{\sim} f_{k-1} \underset{A}{\sim} f_k$$

be a chain of strict annular equivalences. For each $i = 1, \cdots, k$, let α_i be a path in $f_{i-1}(\text{Int } D^n) \cup f_i(\text{Int } D^n)$ which runs from $f_{i-1}(0)$ to $f_i(0)$. Then the path

$$\alpha = \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_k ,$$

which runs from $f_0(0)$ to $f_k(0)$, will be called a *trace* of the given chain of strict annular equivalences.

THEOREM 15.3. *Let f_0 be an arbitrary element of* Hom (D^n, M^n) *and α a closed curve in M^n running through $f_0(0)$. Then α is a trace of a chain of strict annular equivalences beginning and ending with f_0 if and only if α represents an element of the stability subgroup $S\pi_1(M^n)$.*

Let $f_0 \underset{A}{\sim} f_1 \underset{A}{\sim} \cdots \underset{A}{\sim} f_{k-1} \underset{A}{\sim} f_0$ be a chain of strict annular equivalences with trace α. As in the proof of Theorem 3.4, we may assume that

$$f_0(D^n) \supset f_1(D^n) \subset \cdots \supset f_{k-1}(D^n) \subset f_0(D^n) .$$

Then we claim that the local coordinate systems

$$\big(f_0(\text{Int } D^n), f_0\big), \big(f_2(\text{Int } D^n), f_2\big), \cdots, \big(f_{k-2}(\text{Int } D^n), f_{k-2}\big)$$

provide a stable transportation of $(f_0(\text{Int } D^n), f_0)$ around α to itself.

Note first that $\alpha_{2i} \cup \alpha_{2i+1} \subset f_{2i}(\text{Int } D^n)$. Since $f_{2i} \underset{A}{\sim} f_{2i+1}$, $f_{2i+1}^{-1} f_{2i}$ is stable at $f_{2i}^{-1} f_{2i+1}(0)$. Since $f_{2i+1} \underset{A}{\sim} f_{2i+2}$, $f_{2i+2}^{-1} f_{2i+1}$ is stable at $f_{2i+1}^{-1}(0)$. By composition, $f_{2i+2}^{-1} f_{2i}$ is stable at $f_{2i}^{-1} f_{2i+1}(0)$. Since $f_{2i+1}(0)$ lies in $f_{2i}(\text{Int } D^n) \cap f_{2i+2}(\text{Int } D^n)$, the claim is verified and α, according to Theorem 15.2, must represent an element of $S\pi_1(M^n)$.

Now suppose that α represents an element of $S\pi_1(M^n)$. Let $f_0 \underset{A}{\sim} f_1 \underset{A}{\sim} \cdots \underset{A}{\sim} f_{k-1}$ be a chain of strict annular equivalences with trace α, chosen so that $f_{k-1}(D^n) \subset \text{Int} f_0(D^n)$. By associating with this chain a stable transportation of local coordinate systems around α, as in the first part of the proof, we can conclude that $f_0^{-1} f_{k-1}$ is stable at $f_{k-1}^{-1}(0)$. If U is any open n-cell containing $f_0(D^n)$, it follows that f_{k-1} and f_0 are stably, and hence annularly, equivalent in U. But then $f_{k-1} \underset{A}{\sim} f_0$ by [1, Th. 3.5 (i)], and the theorem is proved.

16. Covering spaces

Let $\widetilde{M}^n$ be a covering space over M^n with covering map

$$q\colon \widetilde{M}^n \longrightarrow M^n .$$

We are interested in the relation between $p\colon S(M^n)\rightarrow M^n$ and $\tilde{p}\colon S(\widetilde{M}^n)\rightarrow \widetilde{M}^n$.

Suppose $[\tilde{x}, \widetilde{U}, \tilde{h}]$ is a point of $S(\widetilde{M}^n)$. Let $\widetilde{V}\subset \widetilde{U}$ be a small neighborhood of $\tilde{x}$ which projects homeomorphically under q. Let $\tilde{g} = \tilde{h}/\tilde{h}^{-1}\widetilde{V}$. Then $[\tilde{x}, \widetilde{V}, \tilde{g}] = [\tilde{x}, \widetilde{U}, \tilde{h}]$. Let $x = q(\tilde{x})$, $V = q(\widetilde{V})$ and $g = q\tilde{g}$. Define a map

$$\tilde{q}\colon S(\widetilde{M}^n) \longrightarrow S(M^n)$$

by

$$\tilde{q}\big([\tilde{x}, \widetilde{V}, \tilde{g}]\big) = [x, V, g] .$$

If $[\tilde{x}, \widetilde{V}, \tilde{g}] = [\tilde{x}', \widetilde{V}', \tilde{g}']$, then $\tilde{x} = \tilde{x}'$ and $\tilde{g}'^{-1}\tilde{g}$ must be stable at $\tilde{g}^{-1}(\tilde{x})$. Then $x = q(\tilde{x}) = q(\tilde{x}') = x'$ and $g'^{-1}g = \tilde{g}'^{-1}q^{-1}q\tilde{g} = \tilde{g}'^{-1}\tilde{g}$ in a neighborhood of $g^{-1}(x) = \tilde{g}^{-1}(\tilde{x})$, so $\tilde{q}\big([\tilde{x}, \widetilde{V}, \tilde{g}]\big) = [x, V, g] = [x', V', g',] = \tilde{q}\big([\tilde{x}', \widetilde{V}', \tilde{g}']\big)$. Hence $\tilde{q}$ is well-defined.

Since small local coordinate systems in M^n can be lifted to local coordinate systems in $\widetilde{M}^n$, it is easily seen that $\tilde{q}$ is a covering map, and that the following diagram is commutative.

$$\begin{array}{ccc} S(\widetilde{M}^n) & \xrightarrow{\ \tilde{q}\ } & S(M^n) \\ {\scriptstyle \tilde{}}\downarrow & & \downarrow{\scriptstyle p} \\ \widetilde{M}^n & \xrightarrow[\ q\ ]{} & M^n \end{array}$$

Let m_0 be a basepoint in M^n, and $\tilde{m}_0$ a covering basepoint in $\widetilde{M}^n$. The following theorem asserts that $S(\widetilde{M}^n)$ is minimally determined by the above diagram.

THEOREM 16.1. $q_*S\pi_1(\widetilde{M}^n, \tilde{m}_0) = q_*\pi_1(\widetilde{M}^n, \tilde{m}_0) \cap S\pi_1(M^n, m_0)$.

It follows from the diagram that the left hand side must be contained in the right hand side.

Suppose then that α is a closed curve in M^n based at m_0 and representing an element from the right hand side. Then α is covered by a *closed* curve $\tilde{\alpha}$ in $\widetilde{M}^n$ based at $\tilde{m}_0$. Since α represents an element of $S\pi_1(M^n, m_0)$, it follows from Theorem 15.2 that stable transportation of any local coordinate system around α produces a stably equivalent coordinate system. If the coordinate neighborhoods are small enough, the whole stable transportation can be lifted to $\widetilde{M}^n$. Then again by Theorem 15.2, $\tilde{\alpha}$ must represent an element of $S\pi_1(\tilde{M}^n, \tilde{m}_0,)$, so the right hand side is contained in the left hand side and the theorem is proved.

COROLLARY 1. *Any covering space of a stable manifold is stable.*

COROLLARY 2. *There is a unique minimal stable covering space of any given manifold. It corresponds to the stability subgroup and is equivalent to a component of the sheaf of germs of stable structures on the manifold. Hence for any manifold, the sheaf of germs of stable structures is stable.*

17. Covering transformations

This section is motivated by the following

QUESTION. *If the covering space $\tilde{M}^n$ of M^n is stable, what additional information is needed to deduce the stability of M^n?*

Intuitively, one is led to consider the covering transformations of $\tilde{M}^n$. If these are all stable, it would seem that M^n has a good chance of being stable. However, if the covering is not regular, then the covering transformations are not transitive on fibres, and we are deprived of some information. The following details take this into account.

Let $G = q_*\pi_1(\tilde{M}^n, \tilde{m}_0)$. Then the points of the fibre $q^{-1}(m_0)$ are in one-one correspondence with the right cosets of G in $\pi_1(M^n, m_0)$, with $\tilde{m}_0$ corresponding to G itself. Let $\tilde{m}_\alpha$ denote that point of $q^{-1}(m_0)$ which corresponds to the coset $G\alpha$. Then $\tilde{m}_\alpha = \tilde{m}_\beta$ if and only if $\alpha\beta^{-1} \in G$.

Let $\tilde{f}_0$ be an element of Hom $(D^n, \tilde{M}^n)$ such that

(i) $\tilde{f}_0(0) = \tilde{m}_0$,

(ii) $q/\tilde{f}_0(D^n)$ is a homeomorphism into.

Then $\tilde{f}_\alpha$ will denote the corresponding element of Hom $(D^n, \tilde{M}^n)$ such that

(i) $\tilde{f}_\alpha(0) = \tilde{m}_\alpha$,

(ii) $q\tilde{f}_\alpha = q\tilde{f}_0$.

THEOREM 17.1. $\tilde{f}_0 \underset{s}{\sim} \tilde{f}_\alpha$ *if and only if* $G\alpha$ *meets* $S\pi_1(M^n, m_0)$.

If $\tilde{f}_0 \underset{s}{\sim} \tilde{f}_\alpha$, then there is a chain of strict annular equivalences connecting $\tilde{f}_0$ and $\tilde{f}_\alpha$ with trace $\tilde{\beta}$, which runs from $\tilde{m}_0$ to $\tilde{m}_\alpha$. Let $f = q\tilde{f}_0 = q\tilde{f}_\alpha$. The chain can be chosen with elements small enough so that the whole chain projects down under q to a chain of strict annular equivalences connecting f with itself, and having trace $\beta = q\tilde{\beta}$. Since $\tilde{\beta}$ runs from $\tilde{m}_0$ to $\tilde{m}_\alpha$, β must lie in $G\alpha$. But by Theorem 15.3, β must also lie in $S\pi_1(M^n, m_0)$.

Suppose, on the other hand, that $G\alpha$ meets $S\pi_1(M^n, m_0)$. Without loss of generality, let $\alpha \in S\pi_1(M^n, m_0)$. By Theorem 15.3, there is chain of strict annular equivalences in M^n connecting f with itself and having trace α. Lifting this chain to $\tilde{M}^n$ gives a chain of strict annular equivalences connecting $\tilde{f}_0$ with $\tilde{f}_\alpha$. Hence $\tilde{f}_0 \underset{s}{\sim} \tilde{f}_\alpha$.

COROLLARY 1. *If τ is a covering transformation of $\tilde{M}^n$ taking $\tilde{m}_0$ onto $\tilde{m}_\alpha$, then τ is stable if and only if $G\alpha$ meets $S\pi_1(M^n, m_0)$.*

For τ is stable if and only if $\tilde{f}_0$ is stably equivalent to $\tau\tilde{f}_0 = \tilde{f}_\alpha$.

COROLLARY 2. *If $\tilde{M}^n$ is a covering space of the stable manifold M^n, then every covering transformation is stable.*

COROLLARY 3. *All covering transformations, other than the identity, of a single component of $S(M^n)$ are unstable.*

THEOREM 17.2. *Let $\tilde{M}^n$ be a regular covering space of M^n. Then M^n is stable if and only if $\tilde{M}^n$ is stable and all the covering transformations are stable.*

If M^n is stable, then $\tilde{M}^n$ is stable by Corollary 1 to Theorem 16.1, and all covering transformations are stable by Corollary 2 to Theorem 17.1.

On the other hand, if $\tilde{M}^n$ is stable then $q_*\pi_1(\tilde{M}^n, \tilde{m}_0) = G \subset S\pi_1(M^n, m_0)$. Since the covering is regular, the covering transformations are transitive on fibres. Therefore, by Corollary 1 to Theorem 17.1, every coset of G in $\pi_1(M^n, m_0)$ must meet $S\pi_1(M^n, m_0)$. Hence $\pi_1(M^n, m_0) = S\pi_1(M^n, m_0)$, and M^n is stable.

THEOREM 17.3. *The connected topological manifold M^n is stable if and only if each covering transformation of the universal covering space of M^n can be written as a product of homeomorphisms, each of which is somewhere the identity.*

(IV)

18. Manifolds which admit no stable structure

As remarked at the end of § 10, a stable manifold is automatically orientable. If the annulus conjecture is correct, then it follows from [1, Th. 9.4] that the pseudogroup $SP(R^n)$ of stable coordinate transformations in R^n coincides with the pseudogroup of all orientation preserving coordinate transformations, in which case every orientable manifold is stable. Hence short of a negative solution to the annulus conjecture, we can not exhibit an orientable manifold which admits no stable structure.

Suppose, therefore, that the annulus conjecture is *false*, i.e., that there is a compact region A in R^n bounded by two locally flat $n-1$ spheres S_1 and S_2, which is *not* homeomorphic to $S^{n-1} \times [0, 1]$. Orienting R^n induces orientations of S_1 and S_2 by looking at their bounded complementary domains. Let f be any orientation preserving homeomorphism from S_1 to S_2. Identifying $x \in S_1$ with $f(x) \in S_2$, we obtain from A a decomposition space M^n which is a closed orientable n-manifold.

LEMMA 18.1. *The universal covering space* $\tilde{M}^n$ *of* M^n *is homeomorphic to* $R^n - 0$.

Extend f to an orientation preserving homeomorphism F of R^n onto itself. Then the union of the images of A under the various positive and negative powers of F is a copy of $\tilde{M}^n$. Since this copy lies in R^n, use [1, Th. 6.1] to place a family of stably equivalent locally flat $n-1$ spheres between F^kS_1 and F^kS_2 for each integral value of k. Then $\tilde{M}^n$ is homeomorphic to a doubly infinite chain of spaces, each of which is homeomorphic to $S^{n-1} \times [0, 1]$ by Lemma 9.1 of [1]. Such a space is clearly homeomorphic to $S^{n-1} \times R^1$, which is homeomorphic to $R^n - 0$.

THEOREM 18.2. M^n *does not admit a stable structure.*

Let $\tilde{M}^n = R^n - 0$ and let F be a generator of the infinite cyclic group of covering transformations. If M^n is stable, then F must be stable by Corollary 2 to Theorem 17.1. Hence F extends to a stable homeomorphism of S^n onto itself by Theorem 6.5. But then by [1, Th. 3.5 (i)], A must be homeomorphic to $S^{n-1} \times [0, 1]$, contrary to assumption. Hence M^n can not be stable.

Thus if the annulus conjecture is false in dimension n, there is a closed orientable n-manifold which admits no stable structure. By Theorem 10.4, this manifold can admit neither a differentiable nor a piecewise linear structure. By Theorem 11.2, this manifold can not be triangulated.

It is shown in [6] that if A is a counter-example to the annulus conjecture, then A can not be triangulated.

19. The homogeneity problem

If M_1 and M_2 are homeomorphic stable manifolds, does there exist a *stable* homeomorphism from M_1 onto M_2? Equivalently, is the stable structure on a stable manifold unique up to stable homeomorphism?

As the question is phrased, the answer is *no*, for a stable structure is an oriented structure, and there exist stable manifolds which do not admit orientation reversing homeomorphisms.

Therefore let the stable structures on M_1 and M_2 induce orientations, and assume the existence of an orientation preserving homeomorphism from M_1 onto M_2. Then, does there exist a stable homeomorphism from M_1 onto M_2? Equivalently, are stable structures unique up to orientation?

It is the object of this section to show that the problem of uniqueness of stable structures coincides with the well known homogeneity problem.

Recall that a connected n-dimensional topological manifold M^n is said to be *homogeneous* if, for any two locally flat embeddings f_1, f_2 of D^n into M^n, there is a homeomorphism h of M^n onto itself such that $hf_1 = f_2$. If M^n is orientable and h exists provided f_1 and f_2 induce the same orientation on M^n from a given orientation on D^n, then we say that M^n is *homogeneous up to orientation*.

In other words, homogeneity of M^n means that $H(M^n)$ acts transitively on Hom (D^n, M^n). Note that $H(M^n)$ acts transitively on Hom (D^n, M^n) if and only if $H(M^n)/SH(M^n)$ acts transitively on $\text{Hom}_s(D^n, M^n)$.

Orientable manifolds which admit no orientation reversing homeomorphism cannot be homogeneous, but it is a classical conjecture that all connected manifolds are homogeneous up to orientation. This conjecture has been proved in the differentiable case by Palais [7] and in the piecewise linear case by Newman [8] and Gugenheim [9].

THEOREM 19.1. *Let M^n be a connected stable manifold. Then the stable structure on M^n is unique (unique up to orientation) if and only if M^n is homogeneous (homogeneous up to orientation).*

Assume first that M^n is homogeneous, and let M_1 and M_2 be two stable manifolds with underlying space M^n. Let $h_1: R^n \to U$ be chosen so that (U, h_1) is included in the stable atlas for M_1, according to Theorem 12.1. Similarly, choose $h_2: R^n \to U$ so that (U, h_2) is included in the stable atlas for M_2. Let $f_i = h_i/D^n$ for $i = 1, 2$. Since M^n is homogeneous, there is a homeomorphism h of M^n onto itself such that $hf_1 = f_2$. Then h, viewed as a homeomorphism from M_1 to M_2, is certainly stable at points of $f_1(\text{Int } D^n)$. Since M^n is connected, h is stable, and the stable structure on M^n is unique. Similarly, if M^n is homogeneous up to orientation, then the stable structure on M^n will be unique up to orientation.

Assume now that the stable structure on M^n is unique. Let f_1 and f_2 be elements of Hom (D^n, M^n), and extend f_i to a homeomorphism $h_i: R^n \to U_i \subset M^n$ for $i = 1, 2$. Since U_i is connected, the local coordinate systems (U_1, h_1) and (U_2, h_2) extend to stable atlases, which define stable manifolds M_1 and M_2 on M^n. By assumption, let g be a stable homeomorphism of M_1 onto M_2. To prove that M^n is homogeneous, we will construct a homeomorphism h of M_2 onto itself such that $hgf_1 = f_2$.

By Lemma 3.1, we can assume that $gf_1(D^n) \subset U_2$. Since g is stable, $h_2^{-1}gh_1$ must be stable on Int D^n. Let $h' \in SH_0(R^n)$ agree with $h_2^{-1}gh_1$ on D^n. Then $h_2h'h_2^{-1}$ is a homeomorphism of U_2 onto itself which restricts to the identity near the boundary of U_2.

Let h be the homeomorphism of M_2 onto itself which agrees with

$h_2h'^{-1}h_2^{-1}$ on U_2 and the identity on $M_2 - U_2$. Then on D^n,

$$hgf_1 = h_2h'^{-1}h_2^{-1}gh_1 = h_2h_1^{-1}g^{-1}h_2h_2^{-1}gh_1 = h_2 = f_2 ,$$

so M^n is homogeneous. Similarly, if the stable structure on M^n is unique up to orientation, then M^n is homogeneous up to orientation.

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